Theorem of Tomiyama on Projections of Norm One

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Abstract

We shall give extremely easy and short proofs of the theorem of Tomiyama on projections of norm one.

Usual proofs of the theorem of Tomiyama were very difficult but we give extremely easy and short proofs.

Let A be a C^*-algebra and B a C^*-subalgebra of A. A linear mapping ε of A onto B is said to be a projection of norm one if $\varepsilon(x) = x$ for any $x \in B$ and $\|\varepsilon\| \leq 1$.

Theorem (Tomiyama). *Let ε be a projection of a C^*-algebra A onto a C^*-subalgebra B of norm one. Then it holds that, for any $x \in A$ and $a \in B$,*

(i) *ε is positive;*
(ii) *$\varepsilon(ax) = a\varepsilon(x)$ and $\varepsilon(xa) = \varepsilon(x)a$;*
(iii) *$\varepsilon(x)^*\varepsilon(x) \leq \varepsilon(x^*x)$.*

Proof. We may regard B^{**} as a subalgebra of A^{**}. Then the bitranspose ${}^{tt}\varepsilon$ is a projection of A^{**} onto B^{**} of norm one and an extension of ε. Therefore we may assume that A and B are W^*-algebras. Let 1_A and 1_B be identities of A and B, respectively.

(i) For any $\varphi \in B_+^*$, we have $\|\varphi \circ \varepsilon\| \leq \|\varphi\|$ and

$$\varphi \circ \varepsilon(1_B) = \varphi(1_B) = \|\varphi\|.$$

Hence we have $\varphi \circ \varepsilon \geq 0$. Therefore ε is positive.

(ii) For an element x of A and a projection e of B, put $y = \varepsilon(x(1_A - e))$. It holds that, for any natural number n,

$$\begin{aligned}(n+1)^2\|ye\|^2 &= \|(y + nye)e\|^2 \leq \|y + nye\|^2 \leq \|x(1_A - e) + nye\|^2 \\ &= \|(x(1_A - e) + nye)(x(1_A - e) + nye)^*\| \\ &= \|x(1_A - e)x^* + n^2 yey^*\| \leq \|x\|^2 + n^2\|ye\|^2.\end{aligned}$$

Therefore we obtain $ye = 0$. Replacing x by $x1_B$, we have $\varepsilon(x(1_B - e))e = 0$. Replacing e by $1_B - e$, we have $\varepsilon(xe)(1_B - e) = 0$. Hence we have $\varepsilon(xe) = \varepsilon(xe)e = \varepsilon(x)e$. By spectral decomposition, we obtain $\varepsilon(xa) = \varepsilon(x)a$ for any $a \in B$. Since ε is self-adjoint, we have $\varepsilon(ax) = a\varepsilon(x)$ for any $a \in B$.

(iii) From (i) and (ii), it follows immeadiately that, for any $x \in A$,

$$0 \leq \varepsilon((x - \varepsilon(x))^*(x - \varepsilon(x))) = \varepsilon(x^*x) - \varepsilon(x)^*\varepsilon(x).$$

□

Another Proof. Under the same notations as above, let $ye = v|ye|$ be a polar decomposition of ye; then we have $ve = v$. If $ye \neq 0$, then, for any natural number n, it holds that

$$\|y + nv\| \geq \|v^*(y + nv)e\| = \||ye| + ns(|ye|)\| = \|ye\| + n$$

and

$$\begin{aligned}\|y + nv\|^2 &\leq \|x(1_A - e) + nv\|^2 \\ &= \|(x(1_A - e) + nv)(x(1_A - e) + nv)^*\| \\ &= \|x(1_A - e)x^* + n^2vv^*\| \leq \|x\|^2 + n^2.\end{aligned}$$

This is a contradiction when $2n\|ye\| > \|x\|^2$. Therefore we obtain $ye = 0$. Repeat the above discussions. $\square$

References

[1] J. Tomiyama, *On the projections of norm one in W*-algebras, Proc. Jap. Acad. 33(1957), 608-612.*